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INTERACTION OF NONLINEAR WAVES IN MATERIALS WITH ELASTOPLASTIC BEHAVIOR

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Introduction. As a rule in considering wave problems in condensed materials it is necessary to take account of elastoplastic behavior which may be described by a strongly linear or hysteresis shear stress–shear strain relationship or by a Maxwellian model with nonlinear time for tangential stress relaxation. This behavior is typical not only for traditional physics problems for shock with loading at amplitudes from several to tens of gigapascals [1], but also with quite small amplitudes in metals when the effects of microplasticity are quite marked [2, 3] (so-called 'anomalous nonlinearity' of elastic materials), in polymers with destructive plasticity [3], etc. In the majority of these cases of practical interest wavers may be considered weak in the sense of the smallness of stress in the wave compared with the all-round compression modulus.

In order to solve the problem of wave propagation with a small but finite amplitude in hydrodynamics an effective asymptotic multiscale method has been developed [4, 5] which makes it possible from a complex original system to obtain a nonlinear equation giving a uniform convenient first approximation for solving the original system. A method discovered in [6] by this technique of factorizing the original system into a set of independent nonlinear equations relating to different families of characteristics has made it possible, apart from Burgers turbulence in [6], to consider problems in which there is also marked counter-interaction of waves, for example an acoustic resonator [7] and an elastic layer [8].

In the present work factorization is provided for a nonlinear Maxwell body set of equations which is universal for describing the dynamic behavior of condensed materials with elastoplastic kinetics, and a rough set of independent nonlinear equations is obtained for waves which relate to different families of longitudinal characteristics; waves are connected implicitly through the nonuniform shear of phases. The form of the equation for the phase function which describes interaction as a result of elastoplasticity follows from considering the problem of plane elastic wave propagation over a constant background and in the general case it is prompted by group expressions. Rough sets are obtained for materials which are unbounded in transverse directions and they are suitable for describing traditional shock-wave experiments in a layer (this case is discussed in more detail) and for a thin rod with microplastic deformation kinetics.

The problem of the self-interaction of a plane shock wave with emergence of it at a free surface for a model of an ideally elastoplastic material is resolved analytically. This problem is of considerable practical interest since experiments for measuring the velocity profile for a free [9] or contact [10, 11] surface are fundamental for determining the dynamic properties of metals.

Equations of Motion. Small Parameters. In order to describe the behavior of compact condensed materials with dynamic effects it is possible to use as universal equations for a nonlinear Maxwell body [12]:

$$
\begin{gathered}
\rho(du_i/dt) - \partial\sigma_{ik}/\partial x_k - \partial\sigma_{ik}^{visc}/\partial x_k = 0, \\
d\varepsilon_{ij}/dt = \frac{1}{2}(\delta_{ik} - 2\varepsilon_{ik})(\partial u_k/\partial x_j) + \frac{1}{2}(\delta_{jk} - 2\varepsilon_{jk})(\partial u_k/\partial x_i) + \varphi_{ij}, \\
d\rho/dt + \rho(\partial u_k/\partial x_k) = 0, \\
\rho T(dS/dt) = \partial/\partial x_k[\varkappa(\partial T/\partial x_k)] + \sigma_{im}^{visc}(\partial u_i/\partial x_m) - \frac{1}{2}\rho(\partial E/\partial\varepsilon_{mj})\varphi_{mj},
\end{gathered}
\tag{1}
$$

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$$\sigma_{ik}^{visc} = \eta(\partial u_i/\partial x_k + \partial u_k/\partial x_i) + (\zeta - \frac{2}{3}\eta)(\partial u_j/\partial x_j)\delta_{ik};$$
$$\sigma_{ij} = \rho(\delta_{ik} - 2\varepsilon_{ik})(\partial E/\partial \varepsilon_{kj}). \quad (2)$$

Here σ_{ij} is stress tensor; u_i is material displacement velocity; ε_{ij} is effective elastic strain tensor; φ_{ij} are relaxation terms essential for deformation kinetics; i, j, k, m = 1, 2, 3. Internal energy E for isotropic material is a function of strain tensor invariants and entropy. It is convenient to present this relationship in the form [12] $E = E(\rho, D, \Delta, S)$. The hydrodynamic part of the equation of state, i.e., the dependence of E on density ρ, characterize all-round compression of the volume, and the dependence of effective strain deviator invariants D and Δ characterizes the change in its form:

$$D = \frac{1}{2}(\gamma_{11}^2 + \gamma_{22}^2 + \gamma_{33}^2) + \gamma_{12}\gamma_{21} + \gamma_{23}\gamma_{32} + \gamma_{31}\gamma_{13} + O(\varepsilon^3),\ \Delta =$$
$$\gamma_{11}(\gamma_{22}\gamma_{33} - \gamma_{23}\gamma_{32}) - \gamma_{12}(\gamma_{21}\gamma_{33} - \gamma_{23}\gamma_{31}) + \gamma_{13}(\gamma_{21}\gamma_{32} - \gamma_{31}\gamma_{22}) +$$
$$O(\varepsilon^4),\ \gamma_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}.$$

Small parameter ε is introduced which characterizes the smallness of the relative change in substance density in a wave or the smallness of stress compared with the all-round compression modulus. In Eqs. (1) it is sufficient to limit relaxation terms to the following form: $\varphi_{ij} = -\gamma_{ij}/\tau(\varepsilon_{kl}, S)$ (τ is tangential stress relaxation time). The requirement for describing deformation kinetics in both the elastic and plastic flow region necessitates retention of the markedly nonlinear form of the dependence of τ on strains (or stresses). We recall that in the elastic region dimensionless $\tau' \to \infty$, in the plastic region $\tau' \leq O(1)$, $\tau' = \tau/t_0$, t_0 is characteristic duration of the pressure pulse at the boundary. Formally an ideally elastoplastic model may also be included in this scheme by specific selection of the dependence for φ_{ij} or τ [12].

In this work plane or quasiplane waves are considered which arise with normal impact over the boundary of a half-space or a layer. We assume that the normal to the boundary is directed along axis x_1. In the quasiplane problem the pressure pulse operates on some finite region of the boundary, and small parameter ε_1 is introduced proportional to the square of the ratio of the characteristic wave length to the linear size of this region. Thus it is assumed that in transverse directions [13] $\partial/\partial x_2' \sim \partial/\partial x_3' \sim \varepsilon_1^{1/2}$ ($x_k' = x_k/(C_0 t_0)$). Then from Eqs. (1) it follows that $\gamma_{ij} \sim \varepsilon\varepsilon_1^{1/2}$, $i \neq j$, whereas $\gamma_{11} \sim \gamma_{22} \sim \gamma_{33} \sim \varepsilon$.

Small parameter $\nu = (C_l^2 - C_0^2)/2C_0^2 = 2G/(3\rho_0 C_0^2)$, is introduced, where C_l is the phase velocity of longitudinal elastic waves, C_0 is volumetric sound velocity, and G is shear modulus. For the majority of metals $\nu \leq 0.25$. Experimental data (see e.g. [9, 11]) show that in the range in question pressure parameter ν (similar to Poisson's ratio) changes weakly. Introduction of this small parameter makes it possible to consider the deviator part of the stress tensor as a value of second order smallness, and this means to consider relaxation elastoplastic processes alongside nonlinearity and absorption in the following acoustic approximation. The latter is important for applying the technique of multiscale expansion to Eqs. (1) and reduction of this set to a simpler one.

In modelling intense dynamic loading in the stress deviator there is only retention of terms linear with respect to γ_{ij}, and the equation of state is determined in terms of its dynamic part [1] and the dependence of shear modulus on pressure [14]. By expanding internal energy into a series for the increase in density $\rho' = (\rho - \rho_0)/\rho_0$, entropy $S' = T_0(S - S_0)/C_0^2$, invariants D and Δ, considering that $\partial E/\partial D|_0 = 2G/\rho_0$, from Eqs. (2) retaining in the hydrodynamic part of the tensor σ_{ij} inclusively up to the second order of smallness, we obtain

$$\sigma'_{ii} = \sigma_{ii}/(\rho_0 C_0^2) = -[\rho' + \frac{1}{2}\alpha\rho'^2 + \pi S'/(\rho_0 T_0) +$$
$$O(\varepsilon^3 + \varepsilon^2\nu)] + 3\nu(\gamma_{ii} + O(\varepsilon^2)), \quad (3)$$
$$\sigma'_{ij} = 3\nu[\gamma_{ij} + O(\varepsilon^2\varepsilon_1^{1/2})] + O(\varepsilon^2\varepsilon_1),\ i \neq j,$$

where $E_{\rho s}|_0 = \pi/\rho_0^2$; $\alpha = 4 + \rho_0^3 E_{\rho\rho\rho}|_0/C_0^2$; $T_0 = E_s|_0$ is temperature; index 0 signifies an undisturbed state; parameter α is determined from a two-term equation of state of the Tate and Gruneisen type [1].

Shock Wave Propagation Through Previously Loaded Material. In this section we consider collision of stepped shock waves in the simplest case when it is possible to describe interaction by a changeover from an undisturbed to a previously loaded state ahead of the shock-wave front.

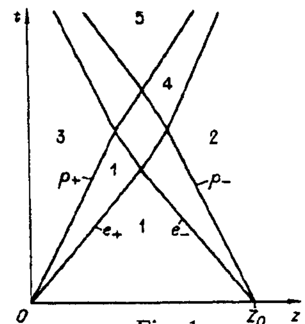

Fig. 1

Let the material be converted by a plane shock wave from a state $(\rho, u, \sigma, \Psi)_r$ with $t' \to -\infty$ into a state $(\rho, u, \sigma, \Psi)_m$ with $t' \to +\infty$; here $\sigma = \sigma_{11}$, $\psi = -\gamma_{11}$, $t' = t/t_0$. We consider an ideally elastoplastic material and Maxwellian material with $0 < \tau' < \infty$. In the first case the state is undisturbed $\Psi_r = 0$, the previously loaded state $\Psi_r = \Psi_*$, and behind the shock-wave front $\psi_m = \psi_*$ ($9/2\ \nu\psi_*$ is yield strength). In the second case in a stationary wave it is evident that $\psi_r = \psi_m = 0$ since $\partial\psi/\partial t'$ and $\partial\rho'/\partial t' \to 0$ with $|t'| \to \infty$.

From the conservation rules in a steady-state jump without considering internal friction viscosity, thermal conductivity, and equation of state (3), it is easy in the approximation adopted to obtain expressions for mass (Lagrangian) velocity M_{rm}, the ratio between velocity and stress, and the dependence $\Psi(\rho')$ in a wave:

$$M_{rm}/(\rho_0 C_0) = \pm\Big[1 + \frac{1}{4}(\alpha + 2)(\rho'_r + \rho'_m) + \frac{3}{2}\nu(\psi_m - \psi_r)/(\rho'_m - \rho'_r)\Big] + O(\varepsilon^2 + \varepsilon\nu),\ \sigma - \sigma_r = -M_{rm}(u - u_r); \tag{4}$$

$$\frac{3}{2}\nu(\psi' - \psi_r) = \delta(\rho' - \rho'_r) - \frac{1}{4}(\alpha + 2)(\rho'^2 - \rho'^2_r) + O(\varepsilon^3 + \varepsilon^2\nu). \tag{5}$$

Here $\delta = 1/2[M_{rm}^2/(\rho_0 C_0)^2 - 1]$. Relationship (5) describes possible states within waves and shows the dependence of shear stress $9/4\ \nu(\psi - \psi_r)$ on true strain ρ'. The parabolic path of this relationship agrees qualitatively for example with changes in shear stress for tungsten [11]. The steady-state solutions for shock waves for obtained as a result of the consistency of Eq. (5) with equations of deformation kinetics.

We calculate the change in Lagrangian movement velocity for steady-state shock waves with head-on collision in an ideally elastoplastic material. Front trajectories are shown in Fig. 1 (along the axes are Lagrangian coordinate and time), where $e_\pm$ are elastic precursor front trajectories, $p_\pm$ are plastic shock-wave trajectories. Here the steady-state propagation regime corresponds singly to elastic and plastic waves which have a stepped form with a uniform value of ρ' behind the front. In the whole complex the elastic + plastic wave is not stationary: as the phase shear propagates between fronts it increases. As is easy to see, this is possible when the amplitude of the plastic wave ρ_m' has the form $2/3\psi_* < \rho_m' < 4\nu/(\alpha + 2)$. States of the material are shown by numbers 0-5 in Fig. 1, but contact discontinuities are not separated since the change in density at the contact discontinuity is a value of the next smallness compared with the change in density in shock jumps. From Eq. (4) taking account of the fact that

$$\rho'_{01} = \rho'_{24} + O(\varepsilon^2),\ \rho'_{13} = \rho'_{45} + O(\varepsilon^2) \text{ and } \rho'_{01} = \frac{3}{2}\psi_*$$

($\rho_{rm}' = \rho_m' - \rho_r'$), we have

$$(M_{24} - M_{01})/(\rho_0 C_0) = \frac{1}{2}(\alpha + 2)\rho'_{02} - \nu,$$
$$(M_{45} - M_{13})/(\rho_0 C_0) = \frac{1}{2}(\alpha + 2)\rho'_{02} \tag{6}$$

etc. It can be seen that with collision of shock waves there is a phase shift not only as a result of hydrodynamic nonlinearity (the first term in the right-hand part), but as a result of the kinetics, i.e. after collision the elastic precursors propagates as a plastic wave.

According to Eq. (4) in a Maxwellian material the velocity of a steady-state shock wave with a change-over to a previously loaded state only changes as a result of nonlinearity since $\psi_m = \psi_r = 0$. In order to reveal the effect of Maxwellian deformation kinetics on the phase shear in a wave profile it is necessary to consider the nonsteady-state problem. It is easy to see that steady-state relationship (5) is a consequence of a nonsteady-state equation

$$\partial\rho'/\partial x_1' - \frac{1}{2}(\alpha + 2)\rho'\partial\rho'/\partial\xi_1 - \frac{3}{2}\nu(\partial\psi/\partial\xi_1) = 0,\ \xi_1 = t' - x_1' \tag{7a}$$

and it is obtained from Eq. (7a) by substituting $(x_1', \xi_1) \to y = \xi_1 + \delta x_1'$ and integrating within the limits from y to $-\infty$. The equation determining deformation kinetics emerges from the second equation of set (1)

$$\partial\psi/\partial\xi_1 = \frac{2}{3}\partial\rho'/\partial\xi_1 + \varphi_{11}(\psi, \rho') + O(\varepsilon^2). \tag{7b}$$

Set (7) describes propagation through a zero background $\rho_r' = 0$ along C_+ of the characteristic direction. The possibility of describing phase shear of interacting waves with a step only occurring due to hydrodynamic nonlinearity for the Burgers equation [6] follows from its invariance with respect to the Galileo transformation. Equation (7a), which may be treated as a generalized Burgers equation, does not exhibit invariance with respect to this transformation:

$$\xi' = \xi_1 + \frac{1}{2}(\alpha + 2)\rho_r' x_1',\ \rho' = \rho'' + \rho_r',\ \psi = \psi' + \psi_r,\ x'' = x_1',\ \rho_r' \neq 0.$$

We assume that after substituting these relationships in Eq. (7) we introduce new variables $\tilde{\xi}$ and $\tilde{\psi}$:

$$\varphi_{11}(\psi' + \psi_r, \rho'' + \rho_r') - \varphi_{11}(\tilde{\psi}, \rho'') = \partial\tilde{\psi}/\partial\tilde{\xi} - \partial\psi'/\partial\xi',$$
$$\tilde{\xi} = \xi' + \nu\theta(x'', \xi'),\ \tilde{x} = x'',$$

then from Eqs. (7) we shall have

$$\partial\rho''/\partial\tilde{x} - \frac{1}{2}(\alpha + 2)\rho''\partial\rho''/\partial\tilde{\xi} - \frac{3}{2}\nu\partial\psi'/\partial\tilde{\xi} +$$
$$\nu(\partial\theta/\partial x'')(\partial\rho''/\partial\tilde{\xi}) + O(\varepsilon^2\nu + \varepsilon\nu^2) = 0.$$

It can be seen that if θ-is selected so that

$$(\partial\theta/\partial x'')(\partial\rho''/\partial\tilde{\xi}) = \frac{3}{2}(\partial\psi'/\partial\tilde{\xi} - \partial\tilde{\psi}/\partial\tilde{\xi}), \tag{8}$$

then we obtain a transformation which with respect to set (7) is invariant. Thus, the effect of Maxwellian deformation kinetics with interaction of a shock wave with a step may be considered in terms of phase shear determined by Eq. (8). It is possible to see that Eq. (8) also describes the shear of phases (6) caused by ideally elastoplastic behavior.

Approximation Nonlinear Steady-State Equations Taking Account of Interaction. Use of the multiscale expansion technique developed in [6] for set of Eqs. (1) and (3) is not a formal procedure and it requires a modified approach. This is connected with the requirement of describing flow in both elastic and plastic regions, i.e. retention of a strongly linear dependence of relaxation terms on strains (or stresses).

In order to factorize sets (1) and (3) we introduce a new unknown function θ_j into the phase variable:

$$\xi_j = t' - \lambda_j^{-1}(x_1' + \varepsilon\Phi_j + \nu\theta_j),\ j = 1,2 \tag{9}$$

($\lambda_j = \pm 1$ corresponds to propagation along $C_\pm$ of characteristic directions). As is well known [6], phase function $\Phi_j(x', t')$ considers interaction of waves j = 1, 2 as a result of quadratic hydrodynamic nonlinearity. Similarly it is possible to demand that θ_j considers interaction as a result of deformation kinetics. Results of the previous section make it possible to do this. In fact, selections of the dependence $\theta_j(x_1', \xi_j)$ in a form similar to Eq. (8) makes it possible to introduce a procedure for using the multiscale expansion technique into the standard procedure.

After determining the phase variable (9) and selecting θ_j in the form of Eq. (12) with the condition $\partial\psi/\partial V \leq O(1)$ in plane (ψ, V) for each Lagrangian particle the factorization procedure for set (1) and (3) is similar [6, 15], and therefore it is given in the appendix.

The final result has the form

$$\lambda_i \partial V_i/\partial z - \frac{1}{4}(\alpha + 2)V_i \partial V_i/\partial \xi_i - 3\nu \partial \psi_i/\partial \xi_i - \frac{1}{2}\mu \partial^2 V_i/\partial \xi_i^2 - \frac{1}{2}\varepsilon_1 \int^{\xi_i} \Delta_\perp V_i d\xi_i' = 0; \tag{10a}$$

$$\partial \psi_i/\partial \xi_i = \frac{1}{3}\partial V_i/\partial \xi_i - \psi_i/\tau'(V_i, \psi_i),\ i = 1,2; \tag{10b}$$

$$\varepsilon \partial \Phi_i/\partial z = -\frac{1}{4}(\alpha + 2)V_j,\ i \neq j,\ i, j = 1,2; \tag{11}$$

$$\partial \theta_i/\partial z = -3[\partial \psi/\partial V - \partial \psi_i/\partial V_i], \tag{12}$$

where

$$\partial \psi_i/\partial V_i = \frac{1}{3} - \psi_i/[(\partial V_i/\partial \xi_i)\, \tau'(V_i, \psi_i)];$$

$$\partial \psi/\partial V = \frac{1}{3} - \psi/[(\partial V/\partial \xi_i)\, \tau'(V, \psi)];$$

$\psi(V)$ is determined from the equation

$$\partial \psi/\partial \xi_i = \frac{1}{3}\partial V/\partial \xi_i - \psi/\tau'(V, \psi); \tag{13}$$

$V_i = -\sigma_{11}'/C' + \lambda_i u_i'$; $V = V_1 + V_2$; $u_1' = u_1/C_0$; $z = x_1'(1 + O(\varepsilon))$ is Lagrangian coordinate over the direction of axis x_1; $C' = C/C_0$; C is Lagrangian phase velocity (14); $\Delta_\perp$ is transverse Laplacian operator (for the asymmetric problem $\Delta_\perp = \partial^2/\partial r'^2 + (1/r')\partial/\partial r'$, $r' = r/r_0$); $\mu \ll 1$ is a dimensionless parameter characterizing internal friction viscosity and thermal conductivity.

In Eq. (12) differentiation with respect to ξ_i with constant z is inferred (in a Lagrangian particle). In Eq. (11) coefficient $1/4(\alpha + 2)$ reflects in the approximation in question a Lagrangian record of set (12). It is noted that with V_j = const expression (12) for θ_i coincides with Eq. (8).

In considering the limitation $\partial\psi/\partial V|_z \leq O(1)$ solution of the set of two independent nonlinear Eqs. (10) for longitudinal waves gives a uniformly convenient, at least at distances $z \leq O[\min(\varepsilon^{-1}, \mu^{-1}, \nu^{-1}, \varepsilon_1{}^{-1})]$, first approximation to solution of the accurate original set (1) and (2). Thus, in order to solve in a quasiacoustic approximation the problem of interaction of counter waves it is sufficient to solve set (12) of independent equations and then by using the solutions obtained for V_k to calculate phase functions from Eqs. (11) and (12). After this $V_i(\xi_i)$ may be corrected by means of strain variable ξ_i (9) to a form which it had before interaction:

$$\xi_i \to \xi_i + \lambda_i^{-1}(\varepsilon\Phi_i(z,\xi_i) + \nu\theta_i(z,\xi_i)).$$

Condition $\partial\psi/\partial V|_z \leq O(1)$ satisfies the elastoplastic model of the material and viscoelastic models in problems in which it is necessary to avoid features at points of inflection $\partial\sigma_{11}/\partial\xi_i = 0$. This viscoelastic model relates for example to a model with elastic unloading. In a number of experiments [9, 11, 16] for studying the propagation of plane compression and rarefaction waves in metals the Lagrangian phase velocity C propagation was measured for fixed levels of strain and stress. In the approximation adopted here

$$C^2 = C_0^2(dz/dt')^2|_{\rho'} = C_0^2\{1 + (\alpha + 2)\rho' + 3\nu(\partial\psi/\partial\rho')|_z\}. \tag{14}$$

Experiments for propagation of shock waves in tungsten and aluminum [11, 16] show a jumpwise change in C^2, and consequently also in $(\partial\psi/\partial\rho')|_z$ at the points of inflection of profile $\partial\sigma_{11}/\partial\xi_i = 0$ and $(\partial\psi/\partial\rho')|_z \simeq O(1)$ close to this point, here $\rho' = 1/2V_i + O(\varepsilon^2 + \varepsilon\nu)$ with $V_j = 0$, $i \neq j$. Therefore limitation $\partial\psi/\partial V|_z \leq O(1)$ is apparently a more rapid limitation in choosing a kinetic deformation model than in the material itself.

Introduction of phase velocity C, which has an entirely specific experimental meaning, makes it possible to combine phase functions in Eq. (9). We designate $F_i = \varepsilon\Phi_i + \nu\theta_i$, then from Eqs. (11), (12), and (14) we have

$$\partial F_i/\partial z = -\frac{1}{2}[C^2(V) - c^2(V_i)]/C_0^2, \tag{15}$$

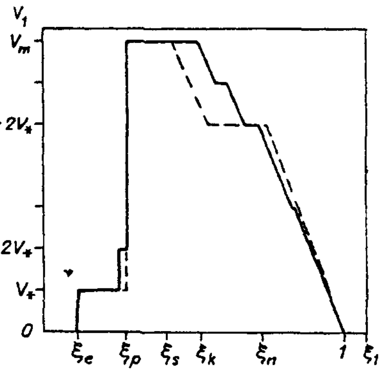

Fig. 2

where C(V) and C(V_j) are Lagrangian phase velocities for the overall wave and the i-th in absence of the j-th respectively.

Collision of Two Plane Stepped Shock Waves. On the basis of the assumptions of Eqs. (9)-(13) we perform solution of the simplest problem of collision of two plane stepped shock waves in an ideal elastoplastic material. In this case Eq. (10b) is written in the form

$$\partial\psi_i/\partial\xi_i = \frac{1}{3}\partial V_i/\partial\xi_i, \text{ when } |\psi_i| < \psi_*, \text{ and } \partial\psi_i/\partial\xi_i = 0, \text{ when } |\psi_i| = \psi_*. \tag{16}$$

Impact occurs simultaneously over boundaries z = 0 and z = Z_0 of the layer (see Fig. 1). Then for limiting large acoustic Reynolds numbers (μ = 0) the solution of Eq. (10) has the form ($V_* \leq V_{im}\, 8\nu/(\alpha + 2)$)

$$V_i = V_*[H(\xi_i - \xi_{i,e}) - H(\xi_i - \xi_{i,p})] + V_{im}H(\xi_i - \xi_{i,p}). \tag{17}$$

Here $\lambda_i = \pm 1$, i = 1, 2; H(x) = 1 with $x \geq 0$; H(x) = 0 with x < 0; $\xi_{i,e}$ and $\xi_{i,p}$ are coordinates of the elastic and plastic fronts: $\xi_{i,e} = [1/8(\alpha + 2)V_* + \nu][-\lambda_i z + 1/2(\lambda_i - 1)Z_0]$, $\xi_{i,p} = 1/8(\alpha + 2)(V_* + V_{im})[-\lambda_i z + 1/2(\lambda_i - 1)Z_0]$ ($V_* = 2\rho_{01}' + O(\varepsilon^2 + \varepsilon\nu)$ is elastic precursor amplitude, V_{im} are wave amplitudes $V_{1m} = 2\rho_{03}' + O(\varepsilon^2 + \varepsilon\nu)$ and $V_{2m} = 2\rho_{02}' + O)\varepsilon^2 + \varepsilon\nu)$). Before collision $\theta_i = \Phi_i = 0$; $\xi_1 = t' - z$, when $\xi_2 < \xi_{2,e}$, and $\xi_2 = t' + z - Z_0$, when $\xi_1 < \xi_{1,e}$. After collision

$$\bar{\xi}_i = [1 - \frac{1}{8}(\alpha + 2)V_{jm} + \frac{1}{2}\nu\Omega_{e,p}(\bar{\xi}_i)]\{[1 + \frac{1}{4}(\alpha + 2)V_{jm} - \nu\Omega_{e,p}(\bar{\xi}_i)]t' -$$

$$\lambda_i z + \frac{1}{2}(\lambda_i - 1)Z_0\} - \frac{1}{2}\frac{1}{4}[\frac{1}{4}(\alpha + 2)V_{jm} - \nu\Omega_{e,p}(\bar{\xi}_i)]Z_0 + O(\varepsilon^2 + \varepsilon\nu + \nu^2),$$

$$\Omega_{e,p}(\bar{\xi}_i) = H(\bar{\xi}_i - \xi_{i,e}) - H(\bar{\xi}_i - \xi_{i,p}).$$

The phase variable after interaction is denoted in terms of $\bar{\xi}_i$. Substitution of $\bar{\xi}_i$ instead of ξ_i in Eq. (17) gives the solution after collision. The change in phase velocity of shock waves as a result of collision agrees entirely with that given in (6).

Self-Reaction of a Π-Shaped Shock Wave on Emerging at a Free Surface. Dynamic Material Properties. We consider a problem of practical interest of the self-reaction of a plane shock wave on emergence at a free surface of a plate obstacle. A wave of rectilinear (Π-shaped) profile forms on entering the obstacle with normal impact with a plate of the same material. If U_0 is striker plate velocity, then shock-wave amplitude $V_{1m} = U_0/C_0$.

We assume that the plate material is ideally elastoplastic. In this case the solution may be written in explicit form. Wave duration at the entry to the obstacle is assumed as unity. Evolution of wave V_1 in the obstacle with large Reynolds numbers is described Eqs. (10a) and (16) with i = 1, $\mu = \varepsilon_1 = 0$. The solution of $V_1(\xi_1, z)$ at depth z > 0 has the form ($V_{1m} < 8\nu/(\alpha + 2)$, i.e. presence of an elastic precursor is assumed)

$$V_1(\xi_1,z) = V_*\Omega_{e,p}(\xi_1) + V_{1m}\Omega_{p,k}(\xi_1) - 2V_*\Omega_{s,k}(\xi_1)(\xi_1 - \xi_s)/(\xi_k - \xi_s) +$$
$$(V_{1m} - 2V_*)[\Omega_{k,n}(\xi_1) + \Omega_{n,l}(\xi_1)(1 - \xi_1)/(1 - \xi_n)], \tag{18}$$

where $\Omega_{gf}(\xi_1) = H(\xi_1 - \xi_g) - H(\xi_1 - \xi_f)$; $\xi_p = -1/8(\alpha + 2)(V_{1m} + V_*)z$; $\xi_e = -|\xi_*| + \xi_p$; $|\xi_*| = (\nu - 1/8(\alpha + 2)V_{1m})\, z > 0$ is elastic precursor duration; $\xi_s = 1 - [1/4(\alpha + 2)V_{1m} + \nu]z$; $\xi_n = 1 - 1/4(\alpha + 2)(V_{1m} = 2V_*)z$; $\xi_k = \xi_n - \nu z$, $\xi_l = 1$. Counter wave $V_2 = 0$, and therefore $V_1 = -2\sigma_{11}'/C'$.

Let the obstacle thickness $z = Z_0/2$; in this plane there is reflection of incident wave V_1 from the free surface and reflected wave V_2 arises so that $(V_1 + V_2)|_{z = z_0/2} = 0$. Free surface velocity $u' = 1/2(V_1 - V_2)|_{z=z_0/2} = V_1(\xi_1, Z_0/2)$ is a value normally measured in experiments. Calculation of self-reaction for the incident wave (i.e., reaction of the incident wave and that caused by its reflection) is carried out by Eqs. (9)-(13) in variables (ξ_1, z) (we consider the case of $4V_* < V_{1m} < 8\nu/(\alpha + 2)$). Before reaction

$$\theta_1 = \Phi_1 = 0,\ 0 < z < \frac{1}{2}(Z_0 - \xi_1 + \xi_e),\ \xi_1 = t' - z.$$

After reaction $(z = Z_0/2)$

$$\varepsilon\Phi_1(\xi_1, \frac{1}{2}Z_0) = \frac{1}{8}(\alpha + 2)V_{1m}(\xi_1 - \xi_p)\Omega_{p,l}(\xi_1) + O(\varepsilon^2 + \varepsilon\nu); \tag{19}$$

$$\theta_1(\xi_1, \frac{1}{2}Z_0) = -\frac{1}{2}|\xi_*|[H(\xi_1 - \xi_p) - H(\xi_1 - \xi_p - \Delta_p)] + \frac{1}{2}(\xi_1 - \xi_p)\Omega_{s,k}(\xi_1) + \frac{1}{2}|\xi_*|\Omega_{a,k}(\xi_1) - \frac{1}{2}(\xi_n - \xi_s)\Omega_{n,b}(\xi_1) - \frac{1}{2}(\xi_k - \xi_s)\Omega_{b,l}(\xi_1). \tag{20}$$

Here $\xi_a = 1/2(\xi_k + \xi_s)$; $\xi_b = \xi_n + \xi_k - \xi_s$; values $|\xi_*|$, ξ_p, ξ_s, ξ_k, ξ_n are in accord with $z = Z_0/2$; $V_1(\xi_p - 0) = V_*$; parameter $0 < \Delta_p \ll 1$ corresponds to the final width of the plastic shock-wave front so that $V_1(\xi_p + \Delta_p) = 2V_*$, $V_1(\xi_p + \Delta_p + 0) = V_{1m}$ with $\Delta_p \to 0$. In Eq. (19) the estimate of accuracy for Φ_1 Eq. (19) complies with $Z_0 \sim O(1)$; the expression for phase function θ_1 describing the 'fine' structure of reaction as a result of elastoplasticity adopted up to $Z_0 \sim O[\min(\varepsilon^{-1}, \nu^{-1})]$.

By means of substituting the independent phase variable $\xi_1 = \xi_1 - \nu\theta_1$, where $\xi_1 = t' - z$, from solving $V_1(\xi_1, Z_0/2)$ by nonuniform deformation of it in accordance with Eq. (20) we obtain the solution $\tilde{V}_1(\xi_1, Z_{0/2})$, in which the effect is considered for self-reaction only as a result of elastoplasticity:

$$\tilde{V}_1(\xi_1, \frac{1}{2}Z_0) = V_*[\Omega_{e,p}(\xi_1) + \Omega_{ee,p}(\xi_1)] + V_{1m}\Omega_{p,ap}(\xi_1) - V_*\Omega_{sp,ap}(\xi_1)(\xi_1 - \xi_{sp})/(\xi_{ap} - \xi_{sp}) + (V_{1m} - V_*)\Omega_{ap,ke}(\xi_1) - V_*\Omega_{ae,ke}(\xi_1) \times (\xi_1 - \xi_{ae})/(\xi_{ke} - \xi_{ae}) + (V_{1m} - 2V_*)\Omega_{ke,bs}(\xi_1) - 2V_*\Omega_{ns,bs}(\xi_1)(\xi_1 - \xi_{ns})/(\xi_{bs} - \xi_{ns}) + (V_{1m} - 4V_*)[\Omega_{bs,lk}(\xi_1) - \Omega_{bk,lk}(\xi_1)(\xi_1 - \xi_{bk})/(\xi_{lk} - \xi_{bk})]. \tag{21}$$

Here

$$\xi_{ee} = \xi_p - \frac{1}{2}\nu|\xi_*|;\ \xi_{sp} = \xi_s + \frac{1}{2}\nu(\xi_s - \xi_p);\ \xi_{ap} = \xi_a + \frac{1}{2}\nu(\xi_{\grave{a}} - \xi_p);\ \xi_{ae} = \xi_a + \frac{1}{2}\nu(\xi_a - \xi_e);\ \xi_{ke} = \xi_k + \frac{1}{2}\nu(\xi_k - \xi_e);\ \xi_{ns} = \xi_n - \frac{1}{2}\nu(\xi_n - \xi_s);\ \xi_{bs} = \xi_b - \frac{1}{2}\nu(\xi_n - \xi_s);\ \xi_{bk} = \xi_b - \frac{1}{2}\nu(\xi_k - \xi_s);\ \xi_{lk} = 1 - \frac{1}{2}\nu(\xi_k - \xi_s).$$

Solution $\tilde{V}_1$ Eq. (21) (solid line) is compared in Fig. 2 with solution $V_1(\xi_1, Z_0/2)$ Eq. (18) (broken line) without considering the effect of self-reaction $(\theta_1 = \Phi_1 = 0)$. It can be seen that self-reaction as a result of elastoplastic deformation kinetics $(\nu \neq 0)$ affects qualitatively the free surface velocity profile; the elastic precursor acquires a two-stage form, and to a considerable extent the step of elastic unloading is smoothed and changes its shape. Appearance of a second step for the precursor, as also for steps in elastic unloading with $V_1 = V_{1m} - V_*$, is caused by reaction with the reflected precursor. The duration of these steps for obstacle thickness $Z_0 \sim O[\min(\xi^{-1}, \nu^{-1})]$ is approximately the same and it is $\nu/2|\xi_*| + O(\varepsilon^2 + \varepsilon\nu)$. Consideration of phase function $\Phi_1(\xi_1, Z_0/2)$ Eq. (19) in the phase variable leads to uniform extension of profile $\tilde{V}_1$ with $\xi_1 > \xi_p$, which does not change qualitatively the nature of the difference between $\tilde{V}_1$ and V_1 (Fig. 2) at the free surface. Appearance of a two-stage precursor in the free surface velocity profile is detected with numerical calculation for the accurate set of equations (type 1)) and the elastoplastic model for beryllium [10, Fig. 7], and tantalum [17, Fig. 7]. In the last case calculation by the accurate set also confirms the presence of steps in elastic unloading with $V_1 = V_{1m} - V_*$ (Fig. 2). Quantitative comparison also gives good conformity.

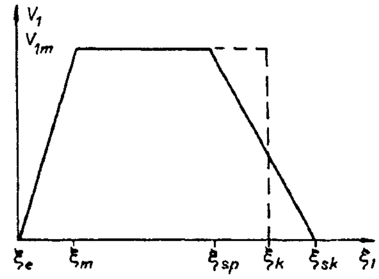

Fig. 3

For completeness we write a solution for a 'rough' structure for self-reaction of a shock wave falling on a free surface which complies with $Z_0 \sim O(1)$. From Eqs. (19) and (20) keeping in $v\theta_1$ only linear terms for small parameters we obtain

$$\xi_1 = (t' - z)[1 - \frac{1}{8}(\alpha + 2)V_{1m}\Omega_{p,l}(\xi_1) - \frac{1}{2}\nu\Omega_{s,k}(\xi_1)] + O(\varepsilon^2 + \varepsilon\nu), \tag{22}$$

$\xi_e < \xi_1 < \xi_l = 1$; ξ_e, ξ_p, ξ_k, ξ_s correspond to $z = Z_{0/2}$. Solution $\tilde{V}_1(\xi_1, Z_{0/2})$ is easily found from Eqs. (18) and (22) in form

$$\tilde{V}_1(\xi_1, \frac{1}{2}Z_0) = V_*\Omega_{e,p}(\xi_1) - 2V_*\Omega_{c,h}(\xi_1)(\xi_1 - \xi_c)/(\xi_k - \xi_s) + V_{1m}\Omega_{p,h}(\xi_1) + (V_{1m} - 2V_*)[\Omega_{h,g}(\xi_1) + \Omega_{gf}(\xi_1)(\xi_f - \xi_1)/(\xi_f - \xi_g)], \tag{23}$$

where $\xi_f = 1 + 1/8(\alpha + 2)V_{1m}$; $\xi_e = \xi_s + 1/2\nu + 1/8(\alpha + 2)V_{1m}$; $\xi_h = \xi_k + \xi_c - \xi_s$; $\xi_g = \xi_n + 1/8(\alpha + 2)V_{1m}$; ξ_n corresponds to $z = Z_{0/2}$. It can be seen that in this case self-reaction as a result of elastoplasticity reduces the duration of the elastic unloading step in $V_1 = V_{1m} - 2V_*$ to the value $\nu/2$, and the elastic unloading slope remains unchanged.

Appearance of a two-stage precursor in the free surface velocity profile is detected in the experiments of Taylor and Rice (Armco-iron), and Hopson (boron carbide, $MgAl_2O_4$) (see [18]). It should be noted that comparison with an experiment in the unloading region is difficult due to superimposition of the coarser effects of failure . As can be seen from numerical modelling of the accurate original set with stage wise complication of the kinetic deformation model carried out in [10], consideration of the dependence of kinetics on deformation velocity smooths the second precursor, and this is probably explained by the fact that is only observed in individual experiments for measuring free (or contact) surface velocity. In fact, as test calculations* showed the same result follows from Eqs. (10) with use of relaxation Eq. (10b), for example for the material of Gilman [19].

It is evident that with mathematical treatment of experiments for measuring free or contact surface velocity the linear approximation which is normally used is not satisfactory. A change-over to the next order in disturbance theory, i.e. to Eqs. (10)-(12), makes it possible to determine dynamic properties more accurately: yield strength, strengthening, metal toughness, etc.

Agreement of the results obtained on the basis of approximation Eqs. (9)-(12) suggested with numerical solutions of the accurate original set and experiments confirms the practical importance of these equations for describing the reaction of nonlinear waves in materials with elastoplastic deformation kinetics.

It is also evident that set (9)-(12) may be used for calculating material flow with spalling in a quasiplanar approximation.

Reaction of Nonlinear Acoustic Waves in Metals with Microplasticity. It is well known that microplasticity leads in metals to anomalously high nonlinearity which is not described within the scope of the five-constant elasticity theory [2]. For definiteness we choose the dependence $\sigma(\varepsilon)$ in the form [20] corresponding to the well-known Granato–Lucca model [2] suggesting absence of residual strains and recovery:

$$\sigma = \tilde{E}(\tilde{\varepsilon} + \gamma f(\tilde{\varepsilon})), \quad f(\tilde{\varepsilon}) = -\mathrm{sgn}\tilde{\varepsilon} \begin{cases} \frac{1}{2}\tilde{\varepsilon}^2, & \tilde{\varepsilon}\tilde{\varepsilon}_t > 0, \\ \frac{1}{2}\varepsilon_m\tilde{\varepsilon}, & \tilde{\varepsilon}\tilde{\varepsilon}_t < 0. \end{cases} \tag{24}$$

*Calculations performed by O. G. Zavileiskii.

Here $\tilde{\varepsilon}$ and ε_m are strain and strain amplitude; $\tilde{E}$ is Young's modulus; $c_y = (\tilde{E}/\rho_0)^{1/2}$ is longitudinal elastic wave velocity in a rod; γ is a parameter defining nonlinearity caused by microplasticity. We consider a thin rod ($\sigma = \sigma_{11}$, $\tilde{\varepsilon} = \varepsilon_{11}$) and $\gamma \gg 1/2(\alpha + 2)$. Characteristic values are $\gamma \cong 10^3$, $\alpha \cong O(1)$, $\tilde{\varepsilon} \cong 10^{-6}$.

In this case the procedure for factorizing Eqs. (1) and (24) leads to an approximation system

$$\begin{gathered}\frac{\partial V_i}{\partial x'} + \lambda_i^{-1}\gamma\frac{\partial f(V_i)}{\partial V_i}\frac{\partial V_i}{\partial \xi_i} = \frac{1}{2}\mu\lambda_i^{-1}\frac{\partial^2 V_i}{\partial \xi_i^2},\\ \varepsilon\frac{\partial \theta_i}{\partial x'} = \gamma\left\{\frac{\partial f(V)}{\partial V}\bigg|_{V=V_1+V_2} - \frac{\partial f(V_i)}{\partial V_i}\right\},\quad i = 1,2,\end{gathered} \tag{25}$$

where $\xi_i = t' - \lambda_i^{-1}(x' + \varepsilon\theta_i)$; $V_i = -\varepsilon_{11} + \lambda_i u_1/c_y$; $\lambda_i = \pm 1$; $x' = x_1/c_y t_0$. Also as for Eqs. (10)-(12) the solution of set (25) gives a uniform approximation suitable for solving original set (1). It is possible to see that if everywhere $\tilde{\varepsilon}\tilde{\varepsilon}_t > 0$, for $\partial f(V)/\partial V = -1/4(V\,\mathrm{sgn}(\tilde{\varepsilon}))$ and θ_i coincides with Φ_1 (11) with an accuracy up to a constant multiple, and set (25) reverts to a set of Burgers equations similar to (6). However, if there is unloading ($\tilde{\varepsilon}\tilde{\varepsilon}_t < 0$), then the situation changes.

By analogy with the previous section we consider the problem of self-reaction of a plane longitudinal wave on its emergence at a free rod end. Shown in Fig. 3 is the response of a free end of a rod with length $\cong O(1)$ calculated by (24) and (25) with action at the other end of a rectangular pulse $V_1 = V_{1m}\Omega_{ek}(\xi_1)$ (the broken line is solution without considering self-reaction, and the solid line is with it). It can be seen that self-reaction caused by microplasticity distorts unloading in the response profile. In Fig. 3

$$\begin{gathered}\xi_{sk} - \xi_k = \xi_k - \xi_{sp} = \frac{1}{16}\gamma V_{1m}(\xi_k - \xi_e) + O(\varepsilon^2),\\ \xi_e(x') = \xi_e(0),\ \xi_k(x') = \xi_k(0) + \frac{1}{8}\gamma V_{1m}x',\ \xi_m(x) = \\ \frac{1}{4}\gamma V_{1m}x' + \xi_e(0).\end{gathered}$$

Equation (25) may evidently be used for studying quasiacoustic resonance in a rod.

Appendix. *Factorization of set (1) and (3).* We expand σ_{ij}, u_j, ρ, ε_{ij} into a series of small parameters ε, ν, μ, ε_1. By drawing attention to the estimates made in the second and fourth sections in the lower order with respect to ε, from Eqs. (1) and (3) we obtain

$$\frac{\partial}{\partial t'}\begin{pmatrix}-\sigma_{11}\\ u'_1\end{pmatrix} + A\frac{\partial}{\partial x'_1}\begin{pmatrix}-\sigma_{11}\\ u'_1\end{pmatrix} = 0,\ A = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{A.1}$$

We introduce matrices T and R consisting of the left- and right-hand eigenvectors of matrix A, then $T_{ik}A_{kl}R_{lj} = \lambda_i\delta_{ij}$, $T_{ik}R_{kj} = \delta_{ij}$, $T = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$, $\lambda_i = (-1)^{i-1}$, $i = 1, 2$. We introduce new variables $V = T\begin{pmatrix}-\sigma_{11}\\ u'_{11}\end{pmatrix}$, ξ_j from Eq. (9), $dz = (1 + \rho')dx_1'$. We expand V and u' into series

$$\begin{gathered}V_i = \varepsilon V_{0i} + \varepsilon^2 V_{1i}^{(\varepsilon)} + \varepsilon\nu V_{1i}^{(\nu)} + \varepsilon\mu V_{1i}^{(\mu)} + \varepsilon\varepsilon_1 V_{1i}^{(\varepsilon_1)} + \ldots,\ u'_l = \varepsilon\varepsilon_1^{1/2}\omega_l + \ldots,\\ i = 1,2,\ l = 2,3.\end{gathered} \tag{A.2}$$

It is clear from (A.1) that V_{0i} is constant along $\xi_i = \mathrm{const}$. In the next series in accordance with the technique of multiscale expansions we require that

$$\begin{gathered}V_{0i} = V_{0i}(\xi_i, z_\varepsilon, z_\nu, z_\mu, z_{\varepsilon_1}, x'_{l\varepsilon}),\\ V_{1i} = V_{1i}(\xi_i, \xi_j, z_\varepsilon, z_\nu, z_\mu, z_{\varepsilon_1}, x'_{l\varepsilon}),\end{gathered} \tag{A.3}$$

where $z_\varepsilon = \varepsilon z$; $z_\nu = \nu z$; $z_\mu = \mu z$; $z_{\varepsilon_1} = \varepsilon_1 z$; $x_{l\varepsilon_1} = \varepsilon_1^{1/2}x'_l$; $l = 2,3.$ Then for the next terms of the expansion we have

$$\begin{gathered}\sum_{j\neq i}(1 - \lambda_i/\lambda_j)\frac{\partial V_{1i}^{(\varepsilon)}}{\partial \xi_j} - \sum_{j\neq i}(\alpha + 2)\sum_m V_{0m}\frac{\partial V_{0j}}{\partial \xi_j} + \left\{\lambda_i\frac{\partial V_{0i}}{\partial z_\varepsilon} - \right.\\ \left.\frac{1}{4}(\alpha + 2)V_{0i}\frac{\partial V_{0i}}{\partial \xi_i}\right\} - \sum_{j\neq i}\left[\lambda_i^{-1}(1 - \lambda_i/\lambda_j)\frac{\partial \Phi_i}{\partial \xi_j} + \frac{1}{4}(\alpha + 2)V_{0j}\right]\frac{\partial V_{0i}}{\partial \xi_i} = 0;\end{gathered} \tag{A.4}$$

$$\sum_{j\neq i}(1-\lambda_i/\lambda_j)\frac{\partial V_{1i}^{(\nu)}}{\partial\xi_j}-3\frac{\partial\psi(V)}{\partial V}\frac{\partial V_{0j}}{\partial\xi_j}+\left\{\lambda_i\frac{\partial V_{0i}}{\partial z_\nu}-3\frac{\partial\psi_i}{\partial\xi_i}\right\}-$$
$$\sum_{j\neq i}\left[\lambda_i^{-1}(1-\lambda_i/\lambda_j)\frac{\partial\theta_i}{\partial\xi_j}+3\left(\frac{\partial\psi(V)}{\partial V}-\frac{\partial\psi_i}{\partial\xi_i}\right)\right]\frac{\partial V_{0i}}{\partial\xi_i}=0,\ i\neq j,\ i,j,m=1,2; \tag{A.5}$$

$$\sum_{j\neq i}(1-\lambda_i/\lambda_j)\frac{\partial V_{1i}^{(\mu)}}{\partial\xi_j}-\frac{1}{2}\sum_{j\neq i}\frac{\mu_{ij}}{\mu}\frac{\partial^2 V_{0j}}{\partial\xi_j^2}+\left\{\lambda_i\frac{\partial V_{0i}}{\partial z_\mu}-\frac{1}{2}\frac{\partial^2 V_{0i}}{\partial\xi_i^2}\right\}=0, \tag{A.6}$$

where $\mu_{ik}=\mu_1(-1)^{i+k}+\pi\mu_2$; $\mu=\mu_{ii}$, $\mu_{1,2}$ are dimensionless coefficients of viscosity and thermal conductivity; π is a material parameter;

$$\sum_m\frac{\partial}{\partial\xi_m}\sum_{j\neq i}(1-\lambda_i/\lambda_j)\frac{\partial V_{1i}^{(\epsilon_1)}}{\partial\xi_j}+\left\{\lambda_i\frac{\partial^2 V_{0i}}{\partial\xi_i\partial z_{\epsilon_1}}-\frac{1}{2}\left(\frac{\partial^2}{\partial x_{2\epsilon_1}^2}+\frac{\partial^2}{\partial x_{3\epsilon_1}^2}\right)V_{0i}\right\}-$$
$$\frac{1}{2}\left(\frac{\partial^2}{\partial x_{2\epsilon_1}^2}+\frac{\partial^2}{\partial x_{3\epsilon_1}^2}\right)V_{0j}=0,\ i\neq j,\ i,j,m=1,2. \tag{A.7}$$

It is necessary that phase functions Φ_i, θ_i in Eqs. (A.4) and (A.5) satisfy Eqs. (11) and (12) taking account of $\partial/\partial\xi_j=[\lambda_i/(1-\lambda_i/\lambda_j)]\partial/\partial z$. Then it is easy to see that the expressions in braces in Eqs. (A.4)-(A.7) on integration with respect to $\xi_j (j\neq i)$ produces secular terms in V_{1i}. In order to avoid secular terms we equate these expressions to zero. Then considering Eq. (A.2) and the expansion

$$\frac{\partial V_{0i}}{\partial z}=\varepsilon\frac{\partial V_{0i}}{\partial z_\varepsilon}+\nu\frac{\partial V_{0i}}{\partial z_\nu}+\mu\frac{\partial V_{0i}}{\partial z_\mu}+\varepsilon_1\frac{\partial V_{0i}}{\partial z_{\varepsilon_1}},$$

we obtain Eq. (10a).

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